

A Simple Proof of Threshold Saturation for Coupled Vector Recursions

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Abstract—Convolutional low-density parity-check (LDPC) codes (or spatially-coupled codes) have now been shown to achieve capacity on binary-input memoryless symmetric channels. The principle behind this surprising result is the threshold-saturation phenomenon, which is defined by the belief-propagation threshold of the spatially-coupled ensemble saturating to a fundamental threshold defined by the uncoupled system.

Previously, the authors demonstrated that potential functions can be used to provide a simple proof of threshold saturation for coupled scalar recursions. In this paper, we present a simple proof of threshold saturation that applies to a wide class of coupled vector recursions. The conditions of the theorem are verified for the density-evolution equations of: (i) joint decoding of irregular LDPC codes for a Slepian-Wolf problem with erasures, (ii) joint decoding of irregular LDPC codes on an erasure multiple-access channel, and (iii) general protograph codes on the BEC. This proves threshold saturation for these systems.

Index Terms—convolutional LDPC codes, spatial coupling, threshold saturation, density evolution, potential functions

I. INTRODUCTION

Low-density parity-check (LDPC) convolutional codes, or spatially-coupled (SC) LDPC codes, were introduced in [1] and observed to have excellent belief-propagation (BP) thresholds in [2], [3], [4]. Recently, they have been observed to approach capacity for a variety of problems [4], [5], [6], [7], [8], [9], [10], [11].

The principle behind their excellent performance is described in [12], where it is shown analytically for the BEC that the BP threshold of a regular SC ensemble converges to the maximum-a-posteriori (MAP) threshold of the uncoupled ensemble. This phenomenon is now called *threshold saturation*. A similar observation was reported independently in [13] and stated as a conjecture. For binary-input memoryless symmetric (BMS) channels, threshold saturation was empirically observed first [4], [5] and then shown analytically [11].

Threshold saturation now appears to be quite general and spatial-coupling has now been applied, with great success, to more general scenarios in information theory and coding [14], [15], [16], [17], [18], [19], [20], [21].

Recently, the authors introduced a simple proof of threshold saturation for coupled scalar recursions where only a few details must be verified for each system [22]. The examples presented therein prove a number of threshold saturation conjectures made in the aforementioned papers (e.g., see [8],

[10]). The proof technique is based on potential functions for density-evolution (DE) recursions and was motivated by the ideas in [23]. Another approach to proving threshold saturation, based on a continuous approximation of spatial-coupling, has appeared in [21] and will appear in [24].

In this paper, the analysis is extended to prove that threshold saturation also occurs for certain coupled systems of vector recursions. In particular, if the single-system vector recursion can be generated by a scalar potential function, then one can also define a scalar potential function for the coupled system. Using this, one can show that threshold saturation occurs for a wider class of problems. For example, this settles conjectures made in [7], [25]. Along with the results in [26], this shows the universality of SC codes for a noisy Slepian-Wolf problem with erasures, when the channels are unknown at the transmitter.

II. A SIMPLE PROOF OF THRESHOLD SATURATION

In this section, we provide a simple proof of threshold saturation via spatial-coupling for a broad class of vector recursions. The main tool is a potential theory for vector recursions that extends naturally to coupled systems.

A. Notation

The following notation is used throughout this paper. We let $d \in \mathbb{N}$ be the dimension for the vector recursion, $\mathcal{X} \triangleq [0, 1]^d$ be the space on which the recursion is defined, and $\mathcal{E} \triangleq [0, 1]$ be the parameter space of the recursive system. For convenience, we let $\mathcal{X}_o \triangleq \mathcal{X} \setminus \{\mathbf{0}\}$ and $\mathcal{E}_o \triangleq \mathcal{E} \setminus \{0\}$. Vectors are denoted in boldface lowercase (e.g. \mathbf{x}, \mathbf{y}), assumed to be row vectors, and partially ordered, $\mathbf{x} \preceq \mathbf{y}$ if $x_i \leq y_i$ for $1 \leq i \leq d$. Standard-weight typeface is used for scalar valued functions with a vector/matrix argument (e.g., $F(\mathbf{x}), F(\mathbf{X})$) and boldface is used to denote a vector valued function with a vector argument (e.g., $\mathbf{f}(\mathbf{x}) = [f_1(\mathbf{x}), \dots, f_d(\mathbf{x})]$). The gradient of a scalar function is defined by $F'(\mathbf{x}) \triangleq [\partial F(\mathbf{x})/\partial x_1, \dots, \partial F(\mathbf{x})/\partial x_d]$ and the Jacobian of a vector function is defined by

$$\mathbf{f}'(\mathbf{x}) = \frac{\partial \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial f_1(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial f_1(\mathbf{x})}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_d(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial f_d(\mathbf{x})}{\partial x_d} \end{bmatrix}.$$

Matrices are denoted in boldface capital letters (e.g. $\mathbf{X} \in \mathcal{X}^n$) and we use $\mathbf{x}_i = [\mathbf{X}]_i$ to denote the i -th row of \mathbf{X} and $x_{i,j} = [\mathbf{X}]_{i,j}$ to denote the (i,j) -th element of \mathbf{X} . Abusing notation, we also allow functions defined for vector arguments

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(e.g., $\mathbf{f}(\mathbf{x})$) to act on matrices (e.g., $\mathbf{f}(\mathbf{X})$) via the rule $[\mathbf{f}(\mathbf{X})]_i = \mathbf{f}(\mathbf{x}_i)$. The notation $\text{vec}(\mathbf{X})$ denotes the transpose of the vector obtained by stacking the columns of \mathbf{X} [27]. The Jacobian and Hessian of a matrix function are denoted by

$$\mathbf{f}'(\mathbf{X}) \triangleq \frac{\partial \text{vec}(\mathbf{f}(\mathbf{X}))}{\partial \text{vec}(\mathbf{X})} \quad \text{and} \quad \mathbf{f}''(\mathbf{x}) \triangleq \frac{\partial \text{vec}(\mathbf{f}'(\mathbf{x}))}{\partial \mathbf{x}}.$$

B. Single-System Potential

First, we define potential functions for a class of vector recursions and discuss the associated thresholds.

Definition 1: A vector admissible system (\mathbf{f}, \mathbf{g}) , governed by the parameter $\varepsilon \in \mathcal{E}$, is defined by the recursion

$$\mathbf{x}^{(\ell+1)} = \mathbf{f}(\mathbf{g}(\mathbf{x}^{(\ell)}); \varepsilon), \quad (1)$$

where $\mathbf{f} = [f_1, \dots, f_d]$ and $\mathbf{g} = [g_1, \dots, g_d]$ are twice continuously differentiable strictly increasing in all arguments (w.r.t the partial order) vector valued functions defined componentwise by $f_i : \mathcal{X} \times \mathcal{E} \rightarrow [0, 1]$ and $g_i : \mathcal{X} \rightarrow [0, 1]$, $1 \leq i \leq d$. Furthermore, $\mathbf{f}(\mathbf{0}; \varepsilon) = \mathbf{f}(\mathbf{x}; 0) = \mathbf{g}(\mathbf{0}) = \mathbf{0}$ and $\mathbf{g}'(\mathbf{x})$ is nonnegative and invertible for $\mathbf{x} \in \mathcal{X} \setminus \{\mathbf{0}, \mathbf{1}\}$.

Definition 2: The potential function $U(\mathbf{x}; \varepsilon)$ of a vector admissible system (\mathbf{f}, \mathbf{g}) is

$$\begin{aligned} U(\mathbf{x}; \varepsilon) &\triangleq \int_0^{\mathbf{x}} [(z - \mathbf{f}(\mathbf{g}(z); \varepsilon)) \mathbf{D} \mathbf{g}'(z)] \cdot d\mathbf{z} \\ &= \mathbf{g}(\mathbf{x}) \mathbf{D} \mathbf{x}^\top - G(\mathbf{x}) - F(\mathbf{g}(\mathbf{x}); \varepsilon), \end{aligned} \quad (2)$$

where $F : \mathcal{X} \times \mathcal{E} \rightarrow \mathbb{R}$ and $G : \mathcal{X} \rightarrow \mathbb{R}$ are functionals that satisfy $F(\mathbf{0}) = 0$, $G(\mathbf{0}) = 0$, $F'(\mathbf{x}) = \mathbf{f}(\mathbf{x}) \mathbf{D}$, and $G'(\mathbf{x}) = \mathbf{g}(\mathbf{x}) \mathbf{D}$, for a positive diagonal matrix $\mathbf{D} \in \mathcal{X}^d$.

Remark 1: The potential function is well-defined because the line integral is path independent. This property is easily verified by taking the derivative with respect to \mathbf{x} . For DE equations, the potential function is closely related to the Bethe free energy (e.g., see [28, Part 2, pp. 62-65]).

Definition 3: For $\mathbf{x} \in \mathcal{X}$, and $\varepsilon \in \mathcal{E}$.

- i) \mathbf{x} is a *fixed point* (f.p.) if $\mathbf{x} = \mathbf{f}(\mathbf{g}(\mathbf{x}); \varepsilon)$.
- ii) \mathbf{x} is a *stationary point* (s.p.) if $U'(\mathbf{x}; \varepsilon) = \mathbf{0}$.

Definition 4: The *fixed-point set*, \mathcal{F} , its \mathbf{x} -support, \mathcal{X}_f , and the *epsilon set*, $\varepsilon(\mathbf{x})$, are given by

$$\begin{aligned} \mathcal{F} &\triangleq \{(\mathbf{x}, \varepsilon) \in \mathcal{X}_0 \times \mathcal{E} \mid \mathbf{x} = \mathbf{f}(\mathbf{g}(\mathbf{x}); \varepsilon)\} \\ \mathcal{X}_f &\triangleq \{\mathbf{x} \in \mathcal{X}_0 \mid \exists \varepsilon \in \mathcal{E} \text{ s.t. } (\mathbf{x}, \varepsilon) \in \mathcal{F}\}, \\ \varepsilon(\mathbf{x}) &\triangleq \{\varepsilon \in \mathcal{E} \mid (\mathbf{x}, \varepsilon) \in \mathcal{F}\}. \end{aligned}$$

Lemma 1: For a vector admissible system, we have

- i) $U(\mathbf{x}; \varepsilon)$ is strictly decreasing in ε , for $\mathbf{x} \in \mathcal{X}_0$ and $\varepsilon \in \mathcal{E}$.
- ii) $\mathbf{x} \in \mathcal{X}$ is a f.p. iff it is a s.p. of the potential.
- iii) For any $\mathbf{x} \in \mathcal{X}_f$, $\varepsilon(\mathbf{x})$ has a single element.

Proof: The potential function is a line integral of $(z - \mathbf{f}(\mathbf{g}(z); \varepsilon)) \mathbf{D} \mathbf{g}'(z)$, and this expression is strictly decreasing in ε , for $\mathbf{x} \in \mathcal{X}_0$ and $\varepsilon \in \mathcal{E}$. For the second property, we observe that the expression is zero iff z is a f.p. because $\mathbf{D} \mathbf{g}'(z)$ is always invertible for $z \in \mathcal{X} \setminus \{\mathbf{0}, \mathbf{1}\}$. For the third, let $\mathbf{x} \in \mathcal{X}_f$. By definition of \mathcal{X}_f , $\varepsilon(\mathbf{x})$ has at least one element. Since $\mathbf{f}(\mathbf{x}; \varepsilon)$ is strictly increasing in ε , for $\mathbf{x} \in \mathcal{X}_0$ and $\varepsilon \in \mathcal{E}$,

$\mathbf{x} = \mathbf{f}(\mathbf{g}(\mathbf{x}); \varepsilon)$ can have at most one ε -root. Therefore, $\varepsilon(\mathbf{x})$ is a singleton and we treat it as a function $\varepsilon : \mathcal{X}_f \rightarrow \mathcal{E}$. ■

Definition 5: Let $\mathbf{x} \in \mathcal{X}$, $\varepsilon \in \mathcal{E}$, and $\mathbf{x}^{(0)} = \mathbf{x}$. Then, $\mathbf{x}^\infty(\mathbf{x}; \varepsilon) = \lim_{\ell \rightarrow \infty} \mathbf{f}(\mathbf{g}(\mathbf{x}^{(\ell)}); \varepsilon)$ (if it exists).

Lemma 2: For all $\varepsilon \in \mathcal{E}$, $\mathbf{x}^\infty(\mathbf{1}; \varepsilon)$ exists.

Proof: Note that $\mathbf{x}^{(1)} = \mathbf{f}(\mathbf{g}(\mathbf{x}^{(0)}); \varepsilon) \preceq \mathbf{1} = \mathbf{x}^{(0)}$ as $\mathbf{1}$ is the greatest element of \mathcal{X} . It follows by induction on ℓ that $\mathbf{x}^{(0)} \succeq \mathbf{x}^{(1)} \succeq \dots \succeq \mathbf{x}^{(\ell)} \succeq \dots \succeq \mathbf{0}$. Hence, the sequence has a limit $\mathbf{x}^{(\infty)} = \mathbf{x}^\infty(\mathbf{1}; \varepsilon)$. ■

Definition 6: The *single-system threshold* is defined to be

$$\varepsilon_s^* \triangleq \sup \{\varepsilon \in \mathcal{E} \mid \mathbf{x}^\infty(\mathbf{1}; \varepsilon) = \mathbf{0}\},$$

and is the ε -threshold for convergence of the recursion to $\mathbf{0}$.

Remark 2: The recursion (1) has no f.p.s in \mathcal{X}_0 iff $\varepsilon < \varepsilon_s^*$. For DE recursions associated with BP decoding, the threshold ε_s^* is called the BP threshold.

Definition 7: The *basin of attraction* for $\mathbf{0}$ is defined by

$$\mathcal{U}_x(\varepsilon) \triangleq \{\mathbf{x} \in \mathcal{X} \mid \mathbf{x}^\infty(\mathbf{x}; \varepsilon) = \mathbf{0}\}.$$

Note that $\mathcal{U}_x(\varepsilon) = \mathcal{X}$ for $\varepsilon < \varepsilon_s^*$ but is a strict subset if $\varepsilon \geq \varepsilon_s^*$.

Definition 8: The *potential threshold* is defined by

$$\varepsilon^* \triangleq \sup \{\varepsilon \in \mathcal{E} \mid \min_{\mathbf{x} \in \mathcal{X}} U(\mathbf{x}; \varepsilon) \geq 0\}. \quad (3)$$

The *energy gap* is, for $\varepsilon > \varepsilon_s^*$, $\Delta E(\varepsilon) \triangleq \inf_{\mathbf{x} \in \mathcal{X} \setminus \mathcal{U}_x(\varepsilon)} U(\mathbf{x}; \varepsilon)$ and $\varepsilon_s^* < \varepsilon < \varepsilon^*$ implies $\Delta E(\varepsilon) > 0$. Since $U(\mathbf{x}; \varepsilon)$ is strictly decreasing in ε , $\varepsilon^* \leq \varepsilon'$ for any ε' that satisfies $\Delta E(\varepsilon') = 0$.

Definition 9 ([29, Conj. 1]): Let the *fixed-point potential*, $Q : \mathcal{X}_f \rightarrow \mathbb{R}$, be defined by $Q(\mathbf{x}) = U(\mathbf{x}; \varepsilon(\mathbf{x}))$. Then, we can define the *Maxwell threshold* ε^{Max} to be

$$\varepsilon^{\text{Max}} \triangleq \inf \{\varepsilon \in \mathcal{E} \mid (\mathbf{x}, \varepsilon) \in \mathcal{F}, Q(\mathbf{x}) = 0\}. \quad (4)$$

Lemma 3: The potential threshold ε^* and the Maxwell threshold ε^{Max} are equal.

Proof: Any fixed-point $\mathbf{x} \in \mathcal{X}_f$ is supported by a unique channel parameter $\varepsilon(\mathbf{x})$. Therefore, for $\mathbf{x} \in \mathcal{X}_f$, we have

$$U(\mathbf{x}; \varepsilon) = F(\mathbf{g}(\mathbf{x}); \varepsilon(\mathbf{x})) - F(\mathbf{g}(\mathbf{x}); \varepsilon) + Q(\mathbf{x}) = Q(\mathbf{x})$$

because $\varepsilon(\mathbf{x}) = \varepsilon$. Let \mathbf{x}^{Max} satisfy $(\mathbf{x}^{\text{Max}}, \varepsilon^{\text{Max}}) \in \mathcal{F}$ and $Q(\mathbf{x}^{\text{Max}}) = 0$. Then, $U(\mathbf{x}^{\text{Max}}; \varepsilon^{\text{Max}}) = Q(\mathbf{x}^{\text{Max}}) = 0$. From Def. 8, we know $\varepsilon^* \leq \varepsilon^{\text{Max}}$. It can also be shown that $\Delta E(\varepsilon^{\text{Max}}) = 0$ and thus, $\varepsilon^{\text{Max}} \leq \varepsilon^*$. This proves equality. ■

III. COUPLED-SYSTEM POTENTIAL

We now extend our definition of potential functions to coupled systems of vector recursions. In particular, we consider a “spatial-coupling” of the single system recursion, (1), that gives rise to the recursion (5) and a closely related matrix recursion (6). For the matrix recursion of the coupled system, we define a potential function and show that, for $\varepsilon < \varepsilon^*$, the only fixed point of the coupled system is the zero matrix.

Definition 10 (cf. [12]): The basic *spatially-coupled vector system* is defined by placing $2L + 1$ \mathbf{f} -systems at positions in the set $\mathcal{L}_f = \{-L, -L + 1, \dots, L\}$ and coupling them with $2L + w$ \mathbf{g} -systems at positions in the set $\mathcal{L}_g = \{-L, -L + 1, \dots, L + (w - 1)\}$. For the coupled system, this leads to the recursion, for $i \in \mathcal{L}_g$, given by

$$\mathbf{x}_i^{(\ell+1)} = \frac{1}{w} \sum_{k=0}^{w-1} \mathbf{f} \left(\frac{1}{w} \sum_{j=0}^{w-1} \mathbf{g}(\mathbf{x}_{i+j-k}^{(\ell)}; \varepsilon_{i-k}) \right), \quad (5)$$

where $\varepsilon_i = \varepsilon$ for $i \in \mathcal{L}_f$, $\varepsilon_i = 0$ for $i \notin \mathcal{L}_f$, $\mathbf{x}_i^{(0)} = \mathbf{1}$ for $i \in \mathcal{L}_g$, and $\mathbf{x}_i^{(\ell)} = \mathbf{0}$ for $i \notin \mathcal{L}_g$ and all ℓ .

Definition 11 (cf. [22]): The recursion in (5) can be rewritten as a *matrix recursion*. Let $\mathbf{X} \in \mathcal{X}^{2L+w}$ have the decomposition $\mathbf{X} = [\mathbf{x}_{-L}^\top, \dots, \mathbf{x}_{L+w-1}^\top]^\top$. Then, (5) is given by

$$\mathbf{X}^{(\ell+1)} = \mathbf{A}^\top \mathbf{f}(\mathbf{A}\mathbf{g}(\mathbf{X}^{(\ell)}; \varepsilon)), \quad (6)$$

where \mathbf{A} is the following $(2L+1) \times (2L+w)$ matrix.

$$\mathbf{A} = \frac{1}{w} \left[\begin{array}{cccccccc} \overbrace{1 \ 1 \ \dots \ 1}^w & 0 & 0 & \dots & 0 \\ 0 & 1 & 1 & \dots & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 1 & \dots & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 \end{array} \right] \left. \vphantom{\begin{array}{c} 1 \\ \vdots \\ 0 \end{array}} \right\} \begin{array}{l} 2L+1 \\ \\ \\ 2L+w \end{array}$$

Definition 12 (cf. [12]): Let $i_0 \triangleq \lfloor \frac{w-1}{2} \rfloor$. The *one-sided spatially-coupled vector system* is a modification of (5) defined by fixing the values of positions outside $\mathcal{L}'_f = \{-L, \dots, i_0\}$. It forces the remaining values to $\mathbf{x}_i^{(\ell)}$, that is $\mathbf{x}_i^{(\ell)} = \mathbf{x}_{i_0}^{(\ell)}$ for $i_0 < i \leq 2L + (w-1)$ and all ℓ .

Lemma 4 (cf. [12, Lem. 14]): For both the basic and one-sided SC systems, the recursions are componentwise decreasing with iteration and converge to well-defined fixed points. The one-sided recursion in Def. 12 is also a componentwise upper bound on the basic SC recursion for $i \in \mathcal{L}_g$ and it converges to a non-decreasing fixed-point vector if $L \geq i_0$.

Sketch of Proof: The proof follows from the monotonicity of \mathbf{f}, \mathbf{g} and a careful treatment of the boundary conditions. ■

Remark 3: This upper bound requires $L \geq i_0 = \lfloor \frac{w-1}{2} \rfloor$, to prevent influence of the right boundary. For the one-sided spatially coupled system with parameter ε and a nontrivial f.p. $\mathbf{X} \neq \mathbf{0}$, the i_0 -th component satisfies $\mathbf{x}_{i_0} \in \mathcal{X} \setminus \mathcal{U}_\varepsilon(\varepsilon)$.

Definition 13: The *coupled-system potential* for general matrix recursions is given by

$$U(\mathbf{X}; \varepsilon) \triangleq \text{Tr}(\mathbf{g}(\mathbf{X})\mathbf{D}\mathbf{X}^\top) - G(\mathbf{X}) - F(\mathbf{A}\mathbf{g}(\mathbf{X}); \varepsilon),$$

where $G(\mathbf{X}) = \sum_i G(\mathbf{x}_i)$ and $F(\mathbf{X}; \varepsilon) = \sum_i F(\mathbf{x}_i; \varepsilon)$.

Remark 4: Using matrix derivatives, one can verify that

$$[U'(\mathbf{X}; \varepsilon)]_i = \left(\mathbf{x}_i - [\mathbf{A}^\top \mathbf{f}(\mathbf{A}\mathbf{g}(\mathbf{X}); \varepsilon)]_i \right) \mathbf{D}\mathbf{g}'(\mathbf{x}_i).$$

Definition 14: The down-shift operator $\mathbf{S}_n : \mathcal{X}^n \rightarrow \mathcal{X}^n$ is defined by $[\mathbf{S}_n \mathbf{X}]_1 = \mathbf{0}$ and $[\mathbf{S}_n \mathbf{X}]_i = \mathbf{x}_{i-1}$ for $2 \leq i \leq n$. In the sequel, the subscript of the down-shift operator is omitted, and it can be inferred from the context.

Lemma 5: Let $\mathbf{X} \in \mathcal{X}^n$ be a matrix with non-decreasing columns generated by averaging the rows of $\mathbf{Z} \in \mathcal{X}^n$ over a sliding window of size w . Then, $\|\text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})\|_\infty \leq \frac{1}{w}$ and $\|\text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})\|_1 = \|\mathbf{x}_n\|_1 = \|\mathbf{X}\|_\infty$.

Proof: For $\|\text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})\|_\infty$, one has

$$|x_{i,j} - x_{i-1,j}| = \left| \frac{1}{w} \sum_{k=0}^{w-1} z_{i+k,j} - \frac{1}{w} \sum_{k=0}^{w-1} z_{i-1+k,j} \right| \leq \frac{1}{w}.$$

Since the columns of \mathbf{X} are non-decreasing, the 1-norm sum telescopes and we get $\|\text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})\|_1 = \|\mathbf{x}_n\|_1 = \|\mathbf{X}\|_\infty$. ■

Lemma 6: For any $\mathbf{X} \in \mathcal{X}^{2L+w}$ that satisfies $[\mathbf{X}]_i = [\mathbf{X}]_{i_0}$ for $i > i_0$, we have $U(\mathbf{S}\mathbf{X}; \varepsilon) - U(\mathbf{X}; \varepsilon) \leq -U(\mathbf{x}_{i_0}; \varepsilon)$.

Proof: First, we rewrite the potential as the summation

$$U(\mathbf{X}; \varepsilon) = \sum_{i=-L}^{L+w-1} [g(\mathbf{x}_i)\mathbf{D}\mathbf{x}_i^\top - G(\mathbf{x}_i) - F([\mathbf{A}\mathbf{g}(\mathbf{X})]_i; \varepsilon)].$$

By virtue of the shift operator, $[\mathbf{A}\mathbf{g}(\mathbf{S}\mathbf{X})]_i = [\mathbf{S}\mathbf{A}\mathbf{g}(\mathbf{X})]_i$. This observation implies that

$$U(\mathbf{S}\mathbf{X}; \varepsilon) - U(\mathbf{X}; \varepsilon) = -F([\mathbf{A}\mathbf{g}(\mathbf{S}\mathbf{X})]_{-L}; \varepsilon) - U(\mathbf{x}_{i_0}; \varepsilon).$$

Since $F(\mathbf{x}; \varepsilon) \geq 0$, this implies the stated result. ■

Lemma 7: For a fixed point of the one-sided SC system \mathbf{X} ,

$$\text{vec}(U'(\mathbf{X}; \varepsilon)) \cdot \text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X}) = 0.$$

Proof: The first $i_0 + L + 1$ rows of $U'(\mathbf{X}; \varepsilon)$ are zero since \mathbf{X} is a fixed point of the one-sided spatially-coupled system. Also, the rows $i_0 < i \leq 2L + (w-1)$ of $\mathbf{S}\mathbf{X} - \mathbf{X}$ are zeros from the right boundary constraint. Hence, the inner product of these two terms is identically zero. ■

Lemma 8: The norm of the Hessian, $U''(\mathbf{X}; \varepsilon)$, of the SC potential is bounded by a constant independent of L and w .

Proof: By direct computation, we obtain

$$\|U''(\mathbf{X}; \varepsilon)\|_\infty \leq \|\mathbf{D}\|_\infty (g'_m + g''_m + 2(g'_m)^2 f'_m) \triangleq K_{f,g},$$

where $g'_m = \sup_{\mathbf{x} \in \mathcal{X}} \|\mathbf{g}'(\mathbf{x})\|_\infty$, $g''_m = \sup_{\mathbf{x} \in \mathcal{X}} \|\mathbf{g}''(\mathbf{x})\|_\infty$ and $f'_m = \sup_{\mathbf{x} \in \mathcal{X}} \|\mathbf{f}'(\mathbf{x}; \varepsilon)\|_\infty$. ■

Theorem 1: For a vector admissible system (\mathbf{f}, \mathbf{g}) , with $\varepsilon < \varepsilon^*$, $w > (dK_{f,g})/(2\Delta E(\varepsilon))$, and $L \geq \lfloor \frac{w-1}{2} \rfloor$, the only fixed point of the spatially-coupled system (Def. 5) is $\mathbf{0}$.

Proof: Let $\varepsilon < \varepsilon^*$. Fix $w > (dK_{f,g})/(2\Delta E(\varepsilon))$ and $L \geq \lfloor \frac{w-1}{2} \rfloor$. Suppose $\mathbf{X} \neq \mathbf{0}$ is the unique fixed point (Lem. 4) of the one-sided recursion in Def. 12. Using Taylor's Theorem, the second-order expansion of $U(\mathbf{S}\mathbf{X}; \varepsilon)$ about \mathbf{X} gives

$$\begin{aligned} & \frac{1}{2} \text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})^\top U''(\mathbf{C}; \varepsilon) \text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X}) \\ &= U(\mathbf{S}\mathbf{X}; \varepsilon) - U(\mathbf{X}; \varepsilon) - \text{vec}(U'(\mathbf{X}; \varepsilon)) \cdot \text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X}) \\ &= U(\mathbf{S}\mathbf{X}; \varepsilon) - U(\mathbf{X}; \varepsilon) \quad (\text{Lem. 7}) \\ &\leq -U(\mathbf{x}_{i_0}; \varepsilon) \leq -\Delta E(\varepsilon), \quad (\text{Lem. 6 and Rem. 3, Def. 8}) \end{aligned}$$

where $\mathbf{C} = \mathbf{X} + t(\mathbf{S}\mathbf{X} - \mathbf{X})$ for some $t \in [0, 1]$. Taking an absolute value and using the bounds in Lemmas 5 and 8 gives

$$\begin{aligned} \Delta E(\varepsilon) &\leq \left| \frac{1}{2} \text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})^\top U''(\mathbf{C}; \varepsilon) \text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X}) \right| \\ &\leq \frac{1}{2} \|\text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})\|_1 \|U''(\mathbf{X}; \varepsilon)\|_\infty \|\text{vec}(\mathbf{S}\mathbf{X} - \mathbf{X})\|_\infty \\ &\leq \frac{1}{2} \|\mathbf{X}\|_\infty K_{f,g} \frac{1}{w} \leq \frac{d}{2w} K_{f,g}. \end{aligned}$$

Therefore $w \leq (dK_{f,g})/(2\Delta E(\varepsilon))$, a contradiction. Thus, the only fixed point for the one-sided spatially-coupled system is the trivial fixed point $\mathbf{0}$. The one-sided spatially-coupled system upper bounds the two-sided system, since $L \geq \lfloor \frac{w-1}{2} \rfloor$. Hence, the only fixed point of the two-sided system is $\mathbf{0}$. ■

IV. APPLICATIONS

In this section, we apply Theorem 1 to a few coding problems that have vector DE recursions. To save space, it relies on definitions and notation from [30], [31], [32].

A. Noisy Slepian-Wolf Problem with Erasures

Two correlated discrete memoryless sources are encoded by two independent linear encoding functions, which are then transmitted through two independent erasure channels with erasure rates ε_1 and ε_2 , respectively. We consider an erasure correlation model between the two sources. More specifically, let Z be a Bernoulli- p random variable such the two sources are the same Bernoulli- $\frac{1}{2}$ random variable if $Z = 1$ and are i.i.d. Bernoulli- $\frac{1}{2}$ random variables if $Z = 0$. The decoder is assumed to have access to the side information Z .

Assume that the i -th source sequence is mapped into an LDPC code with degree distribution (d.d.) (λ_i, ρ_i) and design rate $\gamma = 1 - L'_i(1)/R'_i(1)$ using a punctured systematic encoder of rate $\gamma/(1-\gamma)$. The fraction of punctured systematic bits is γ (see [7] for details). Let $x_1^{(\ell)}$ (resp. $x_2^{(\ell)}$) be the average erasure rate of messages, from bit nodes to check nodes, corresponding to source 1 (resp. 2) and $\mathbf{x}^{(\ell)} = [x_1^{(\ell)}, x_2^{(\ell)}]$. Let $\mathcal{C} : [0, 1] \rightarrow [0, 1]^2$, $\varepsilon \mapsto [\varepsilon_1(\varepsilon), \varepsilon_2(\varepsilon)]$ with $[\varepsilon_1(0), \varepsilon_2(0)] = \mathbf{0}$, be continuous and monotonically increasing. The DE recursion in [7] is easily generalized to asymmetric d.d.s and can be written in the form of (1) with

$$\begin{aligned}\psi(x; \varepsilon) &\triangleq (1 - \gamma)\varepsilon + \gamma(1 - p + px), \\ \mathbf{f}(\mathbf{x}; \varepsilon) &\triangleq [\psi(L_2(x_2); \varepsilon_1(\varepsilon))\lambda_1(x_1), \psi(L_1(x_1); \varepsilon_2(\varepsilon))\lambda_2(x_2)], \\ \mathbf{g}(\mathbf{x}) &\triangleq [1 - \rho_1(1 - x_1), 1 - \rho_2(1 - x_2)].\end{aligned}$$

Using Def. 2 and $\mathbf{D} = \text{diag}(L'_1(1), L'_2(1))$, one finds that

$$\begin{aligned}F(\mathbf{x}; \varepsilon) &= \psi(L_1(x_1); \varepsilon_2(\varepsilon))L_2(x_2) \\ &\quad + \psi(L_2(x_2); \varepsilon_1(\varepsilon))L_1(x_1) - \gamma p L_1(x_1)L_2(x_2), \\ G(\mathbf{x}) &= \sum_{k=1}^2 L'_k(1) \left(x_k + \frac{R_k(1 - x_k) - 1}{R'_k(1)} \right).\end{aligned}$$

For asymmetric d.d.s, one can also generalize both the trial entropy $P(\mathbf{x})$, from [26, Lem. 4], and the mapping $\varepsilon(\mathbf{x}) = [\varepsilon^{[1]}(\mathbf{x}), \varepsilon^{[2]}(\mathbf{x})]$, from [26, Sec. II-A]. This gives

$$U(\mathbf{x}; \varepsilon) = (1 - \gamma) ((\varepsilon(\mathbf{x}) - [\varepsilon_1(\varepsilon), \varepsilon_2(\varepsilon)])\mathbf{L}(\mathbf{g}(\mathbf{x}))^\top - P(\mathbf{x})),$$

where $\mathbf{L}(\mathbf{g}(\mathbf{x})) = [L_1(g_1(x_1)), L_2(g_2(x_2))]$. Substituting $\varepsilon \mapsto \varepsilon(\mathbf{x})$ implies that $Q(\mathbf{x}) = -(1 - \gamma)P(\mathbf{x})$. Therefore, we find that the Maxwell threshold from Def. 9 is equivalent to the standard Maxwell threshold [29, Conj. 1].

Corollary 1: Applying Theorem 1 shows that, if $\varepsilon < \varepsilon^{\text{Max}}$ and $w > K_{f,g}/\Delta E(\varepsilon)$, then the SC Slepian-Wolf DE recursion must converge to the zero matrix.

Remark 5: For special cases, one can use the methods in [26], [31], [32] to show that the Maxwell threshold defined above is an upper bound on the MAP threshold. These references also show that, for regular LDPC codes with fixed rate and increasing degrees, the upper bound approaches the information-theoretic limit. Therefore, SC regular LDPC codes are universal (e.g., see [7]) for this problem.

B. Erasure Multiple-Access Channel

We consider the two-user MAC channel with erasure noise (EMAC) from [25]. Let the inputs be $X^{[1]}, X^{[2]} \in \{\pm 1\}$ and

the output be

$$Y = \begin{cases} X^{[1]} + X^{[2]} & \text{with probability } 1 - \varepsilon, \\ ? & \text{with probability } \varepsilon \end{cases}.$$

Assume that the source sequences are encoded by LDPC codes with d.d.s (λ_1, ρ_1) and (λ_2, ρ_2) . Let $x_1^{(\ell)}$ (resp. $x_2^{(\ell)}$) be the average erasure rate of messages from bit nodes to check nodes corresponding to user 1 (resp. 2) and $\mathbf{x}^{(\ell)} = [x_1^{(\ell)}, x_2^{(\ell)}]$. In [26], the DE recursion is written as (1), with

$$\begin{aligned}\psi(x; \varepsilon) &\triangleq \varepsilon + (1 - \varepsilon)x/2, \\ \mathbf{f}(\mathbf{x}; \varepsilon) &\triangleq [\psi(L_2(x_2); \varepsilon)\lambda_1(x_1), \psi(L_1(x_1); \varepsilon)\lambda_2(x_2)], \\ \mathbf{g}(\mathbf{x}) &\triangleq [1 - \rho_1(1 - x_1), 1 - \rho_2(1 - x_2)].\end{aligned}$$

Using Def. 2 and $\mathbf{D} = \text{diag}(L'_1(1), L'_2(1))$, one finds that

$$\begin{aligned}F(\mathbf{x}; \varepsilon) &= \varepsilon[L_1(x_1) + L_2(x_2)] + (1 - \varepsilon)L_1(x_1)L_2(x_2)/2, \\ G(\mathbf{x}) &= \sum_{k=1}^2 L'_k(1) \left(x_k + \frac{R_k(1 - x_k) - 1}{R'_k(1)} \right).\end{aligned}$$

Let the trial entropy, $P(\mathbf{x})$, and the $\varepsilon(\mathbf{x})$ be defined by [26, Lem. 10]. Then, $U(\mathbf{x}; \varepsilon)$ equals

$$(\varepsilon(\mathbf{x}) - \varepsilon)[\mathbf{L}(\mathbf{g}(\mathbf{x}))\mathbf{1}^\top - \frac{1}{2}L_1(g_1(x_1))L_2(g_2(x_2))] - P(\mathbf{x}),$$

and substituting $\varepsilon \mapsto \varepsilon(\mathbf{x})$ implies that $Q(\mathbf{x}) = -P(\mathbf{x})$. Therefore, we find that the Maxwell threshold from Def. 9 is equivalent to the standard Maxwell threshold [29, Conj. 1].

Corollary 2: Applying Theorem 1 shows that, if $\varepsilon < \varepsilon^{\text{Max}}$ and $w > K_{f,g}/\Delta E(\varepsilon)$, then the SC DE recursion for the erasure MAC channel must converge to the zero matrix.

Remark 6: For special cases, one can use the methods in [26], [31], [32] to show that the Maxwell threshold defined above is an upper bound on the MAP threshold. These references also show that, for regular LDPC codes with fixed rate and increasing degrees, the upper bound approaches the information-theoretic limit of the EMAC channel.

C. General Protograph Codes on the BEC

Consider the protograph ensemble [33] defined by an $m \times n$ protograph parity-check matrix H (e.g., $H = [3 \ 3]$ defines a (3,6)-regular code) and let $[k]$ denote the set $\{1, 2, \dots, k\}$. Let the dimension of the recursion, d , equal the number of non-zero entries in H and let the functions $r : [d] \rightarrow [m]$, $c : [d] \rightarrow [n]$, and $e : [d] \rightarrow \{1, 2, \dots\}$ map the index of each non-zero entry to its row, column, and value (i.e., $e(k) = H_{r(k), c(k)}$ for $k \in [d]$). Let $\varepsilon_j(\varepsilon)$ be the erasure probability of the j -th bit node in the protograph as a function of the channel parameter ε . Then, the bit- and check-node DE update functions $\mathbf{f}(\mathbf{x}), \mathbf{g}(\mathbf{x})$ are given by

$$f_k(\mathbf{x}; \varepsilon) = \varepsilon_{c(k)}(\varepsilon) \prod_{i \in [d]: c(i)=c(k)} x_i^{e(i)-\delta_{i,k}}, \quad g_k(\mathbf{x}) = 1 - \prod_{j \in [d]: r(j)=r(k)} (1 - x_j)^{e(j)-\delta_{i,k}},$$

where $\delta_{i,j}$ is the Kronecker delta function. From this, we make an educated guess that the bit- and check-node potentials are

$$\begin{aligned}F(\mathbf{x}; \varepsilon) &= \sum_{j=1}^n \varepsilon_j(\varepsilon) \prod_{i \in [d]: c(i)=j} x_i^{e(i)} \\ G(\mathbf{x}) &= \sum_{k=1}^d e(k)x_k - \sum_{i=1}^m \left[1 - \prod_{j \in [d]: r(j)=i} (1 - x_j)^{e(j)} \right].\end{aligned}$$

Since each non-zero entry in H appears in only one row and one column, it is easy to verify that

$$\frac{d}{dx_k} F(\mathbf{x}; \varepsilon) = e(k) f_k(\mathbf{x}; \varepsilon) \quad \text{and} \quad \frac{d}{dx_k} G(\mathbf{x}) = e(k) g_k(\mathbf{x}).$$

This shows that one can choose $\mathbf{D} = \text{diag}(e(1), \dots, e(d))$ and then apply Def. 2 to define a potential function for the protograph DE update. For the protograph ensemble, we conjecture that the fixed-point potential, $Q(\mathbf{x})$, will also be a scalar multiple of the trial entropy defined by integration of the BP EXIT curve [30].

V. CONCLUSIONS

Based on the work in [22], a new theorem is presented that provides a simple proof of threshold saturation for a broad class of vector recursions. The conditions of the theorem are verified for the density-evolution equations associated with: (i) irregular LDPC codes for a Slepian-Wolf problem with erasures, (ii) irregular LDPC codes on the erasure multiple-access channel, and (iii) general protograph codes on BEC. This provides the first proof of threshold saturation for these systems. Along with the results in [26], [31], [32], this also shows that SC codes are universal (e.g., see [7]) for the noisy Slepian-Wolf problem with erasures.

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